Eikonal Theory; Supplement

Recall:

- for Helmholtz Eqn, derived:

$$(\underline{\nabla}\phi)^2 = \frac{\omega^2}{c(\underline{x})^2} \Rightarrow \text{eikonal eqn.}$$

↳ inhomogeneous speed.

$$\psi \sim A e^{i\phi}$$

- as rays ⊥ phase fronts

$$\underline{\nabla}\phi \equiv \underline{k} = \underset{\text{aka WKB}}{\underbrace{k(\underline{x})}}$$

$$\omega = -\frac{\partial\phi}{\partial t}$$

so

$$\psi = A\exp\Big[i\overbrace{\Big(\int \underline{k}(\underline{x})\cdot d\underline{x} - \omega t\Big)}^{\Phi}\Big]$$

eikonal approx. to wave function

- now, for ray trajectories, observe total phase Φ

$$d\Phi = \underline{k}\cdot d\underline{x} - \omega\, dt$$

$$= \left(\underline{k}\cdot\frac{d\underline{x}}{dt} - \omega\right) dt$$

$S \leftrightarrow \Phi$ is key analogy

analogous

$$S = \int L\, dt \Rightarrow dS = L\, dt$$
$$= (p\dot{q} - H)\, dt$$

$\therefore$

— obvious analogy

$\underline{k} \leftrightarrow \underline{p}$ i.e. QM: $\underline{p} = \hbar \underline{k}$

$\underline{x} \leftrightarrow \underline{q}$ $E = \hbar\omega$

$\omega \leftrightarrow H$

so $\dfrac{d\underline{k}}{dt} = -\dfrac{\partial\omega}{\partial\underline{x}} \Leftrightarrow \dfrac{d\underline{p}}{dt} = -\dfrac{\partial H}{\partial\underline{q}}$

$\dfrac{d\underline{p}}{dt} = -\partial H/\partial\underline{q}$

$\dfrac{d\underline{x}}{dt} = \dfrac{\partial\omega}{\partial\underline{k}} \Leftrightarrow \dfrac{d\underline{q}}{dt} = \dfrac{\partial H}{\partial\underline{p}}$

$\dfrac{dq}{dt} = \dfrac{\partial H}{\partial p}$

as in terms $c(\underline{x})$:

$$\omega^2 = c(x)^2 k^2$$

$$\frac{d\underline{k}}{dt} = -k \frac{\partial c(\underline{x})}{\partial \underline{x}}$$

$$\frac{d\underline{x}}{dt} = c(\underline{x})\,\hat{k} .$$

N.B.

→ $\partial\omega/\partial\underline{k} \equiv \underline{v}_{gr}$ group velocity

What does $\underline{v}_{gr}$ mean?

Consider wave packet,

carrier $\underline{k}_0$
spread $\Delta\underline{k}$

$$\phi\Big|_{t=0} \sim \underbrace{e^{i\underline{k}_0\cdot\underline{x}}}_{\text{carrier}}\ \underbrace{F(\underline{x})}_{\text{envelope}}$$

$$F(\underline{x}) \sim \sum_{\Delta\underline{k}} e^{i\Delta\underline{k}\cdot\underline{x}}$$

so

$$\phi(\underline{x},t) \sim \sum_{\Delta\underline{k}} e^{i\left[(\underline{k}_0+\Delta\underline{k})\cdot\underline{x} - \omega(\underline{k}_0+\Delta\underline{k})t\right]}$$

carrier

$$\sim e^{i(\underline{k}_0\cdot\underline{x} - \omega(\underline{k}_0)t)} \sum_{\Delta\underline{k}} e^{i\Delta\underline{k}\cdot\underline{x}}\, e^{-\frac{\partial\omega}{\partial\underline{k}}\cdot\Delta\underline{k}\,t}$$

$$\phi(\underline{x},t) \sim e^{i(\underline{k_0}\cdot\underline{x}-\omega t)}\, F\Big(\underline{x} - \underbrace{\frac{\partial\omega}{\partial \underline{k}}}_{\underline{v}_{gr}} t\Big)$$

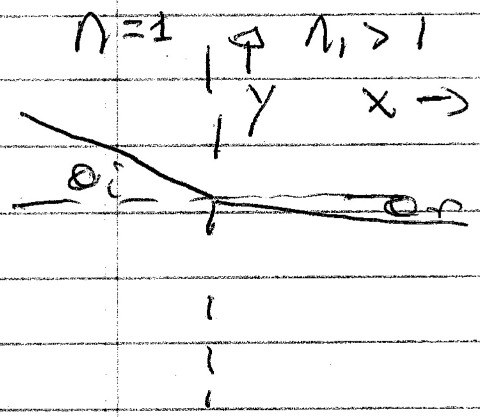
$\Rightarrow$ rate/speed at which energy propagated.

N.B. $E_{pckt} \sim |\phi_k|^2 \sim |F|^2$

$\Rightarrow$ V_{gr} sets speed at which energy propagates.

$$\Rightarrow \quad \frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}} \qquad \Rightarrow \text{Snell's Law.}$$

i.e.



$$\frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial\underline{x}} \qquad \Rightarrow \quad \frac{dk_y}{dt} = 0$$

$$k_{y_-} = k_{y_+} \qquad \Rightarrow \quad k_- \sin\theta_i = k_+ \sin\theta_r$$

$$k_-^2 = n_0^2\,\frac{\omega^2}{c_0^2} \qquad k_+^2 = n_1^2\omega^2/c_0^2$$

$n_0 \sin\theta_i = n_i \sin\theta_r$ ✓

- Now, if seek first principles approach

$\Rightarrow$ extremize Φ (i.e. look for phase stationarity)

stationarity $\rightarrow$ trajectories i.e. ray or particle (capt. full wave)

$$\delta\Phi = \delta\int\left[\underline{k}\cdot d\underline{x} - \omega\, dt\right]$$

$$= \delta\int\left[\underline{k}\cdot\dot{\underline{x}} - \omega\right]dt$$

$\left(\begin{array}{l} t \to i\tau \\ \text{Steepest Descents} \\ \int d\underline{k}\, e^{i\Phi} \to \underline{\text{packet}} \end{array}\right)$

$$= \int\left[\delta\underline{k}\cdot\dot{\underline{x}} + \underline{k}\cdot\delta\dot{\underline{x}} - \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right]dt$$

but $\delta\dot{\underline{x}} = \dfrac{d}{dt}\delta\underline{x}$

e.p. fixed.

$$\delta\Phi = \cancel{\underline{k}\cdot\delta\underline{x}\Big|_{t_1}^{t_2}} + \int\left[\delta\underline{k}\cdot\dot{\underline{x}} - \frac{d\underline{k}}{dt}\cdot\delta\underline{x}\right.$$

$$\left. - \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right]$$

$\delta \Phi \Rightarrow$

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial k} , \qquad \frac{dk}{dt} = -\frac{\partial \omega}{\partial x} .$$

$\Rightarrow$ Liouville Thm $\Rightarrow$ Wave Kinetics.

N.B.: for semi-classical limit

$$\underline{p} = N\hbar \underline{k}$$

$$E = N\hbar\omega$$

$N \Leftrightarrow \rho$.

Finally, : to recover Fermat, note:

$$\delta \Phi = 0$$

$$d\phi = \underline{k} \cdot d\underline{x} - \omega t$$

so for ray path:

$$\delta \Phi_s = 0, \qquad \Phi_s = \int \underline{k} \cdot d\underline{x}$$

but

$$\underline{k} \cdot d\underline{x} = \underline{k} \cdot \frac{d\underline{x}}{ds}\, ds \Rightarrow$$

$$\underline{k} = \underline{\nabla}\phi = |\underline{\nabla}\phi|\, \underline{t}$$

$$d\underline{x}/ds = \underline{t}$$

so

$$\delta \Phi_s = \delta \int |\nabla\phi|\, ds$$

but $|\nabla\phi|^2 = \omega^2/c^2 = \frac{\omega^2}{c_0^2} n(\underline{x})^2$

$\Rightarrow$

$$\delta \Phi_s = \delta \frac{\omega}{c_0} \int ds\, n(\underline{x}) = 0$$

$$\Rightarrow \delta \int ds\, n(\underline{x}) = 0 \quad \checkmark$$